Notes on Addictive Decomposition of Polynomials

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The purpose of this note is to study works on addictive decompositions of polynomials in [Kle13] . At least we have a positive answer to my question "Uniqueness of decomposition into summands of homogeneous polynomials of degree$\geq$ 2" on StackExchange , maybe except positive characteristic case. Moreover, I put some simple propositions at last based on the results.

1 Basic Definitions

Fix[1] a field k with arbitrary characteristic. We use the notion of so-called **power divided algebra**. For explicit definition see Divided power structure in Wikipedia. Here we use the divided power algebra $\mathcal{R}=\bigoplus_{d=0}^{\infty}\mathcal{R}_d=k[x_1,\ldots,x_r]^{DP}$, which is usually denoted $k\langle x_1,\ldots,x_r\rangle$. It can be defined as a graded (commutative) k-algebra

$$k\left[x_i^{(n)}\right]_{1\leq i\leq r,n\in\mathbb{Z}_{>0}}\Big/\Big/\left\langle x_i^{(m)}x_i^{(n)}=\binom{m+n}{m}x_i^{(m+n)}\right\rangle,$$

where $x_i^{(n)}$ is of degree n. $(-)^{(n)}$ can be extended to the whole part of ideal of homogeneous elements of degree$\geq$ 1 by letting $x^{(0)}=1$, $(cx)^{(n)}=c^nx^{(n)}$, $(x+y)^{(n)}=\sum_{k=0}^{n}x^{(k)}y^{(n-k)}$ and $\left(x^{(n)}\right)^{(m)}=\frac{(nm)!}{(n!)^m m!}x^{(mn)}$. When $\operatorname{char}k=0$ this recovers $k[x_1,\ldots,x_r]$.

Note that $R=\bigoplus_{d=0}^{\infty}R_d=k[\partial_1,\ldots,\partial_r]$ acts on $\mathcal{R}$ by $\partial_i x_j^{(n)}=\delta_{ij}x_j^{(n-1)}$. Indeed, $\mathcal{R}$ can be viewed as the degree-wise dual as a k-linear space of R.

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[1] If you just care about characteristic 0, you can ignore and jump to "Given a homogeneous" paragraph. Just remember $\mathcal{R}$ is just $k[x_1,\ldots,x_r]$, R is just $k[\partial_1,\ldots,\partial_r]$ and $x^{(\alpha)}=\frac{x^\alpha}{\alpha!}$ where α is a multi-index.

Given a homogeneous polynomial f of degree d in $\mathcal{R}$, we denote the ideal of R annihilating f by $\operatorname{Ann}_R(f)$, which is graded along the grading of R.[2] Note that the pair $R_d \times \mathcal{R}_d \to k$ is non-degenerate.

For $P \in \mathrm{GL}_r(k)$, one can define k-algebra endomorphisms, to say

$$\begin{array}{cccc} \phi_P: & \mathcal{R} & \to & \mathcal{R}, \\ & (x_i)_i & \mapsto & \sum_j P_{ji} x_j \end{array}$$

and the dual map

$$\begin{array}{cccc} \phi_P: & R & \to & R, \\ & \partial & \mapsto & P^{-1}\partial \end{array}$$

both denoted by ϕ_P. Then $\phi_P(Df) = (\phi_P D)(\phi_P f)$, and $\operatorname{Ann}_R(\phi_P f) = \phi_P(\operatorname{Ann}_R f)$.

2 Addictive Splitting

There is a more intrinsic definition for independence in variables:

Definition 2.1. For $g_i \in \mathcal{R}_{d_i}, 1 \le i \le n$, we say they are **in independent variables** if

$$R_{d_i-1}(g_i) \cap \left(\sum_{j \neq i} R_{d_j-1}(g_j) \right) = 0 \subset \mathcal{R}_1, \forall i.$$

It is natural to think of $R_{d-1}(f)$ as naive variables of $f \in \mathcal{R}_d$. Note that $R_{d_i-1}(g_i) \cap \left(\sum_{j\neq i} R_{d_j-1}(g_j) \right) = 0, \forall i$ would imply for $e > 0$ that $R_{d_i-e}(g_i) \cap \left(\sum_{j\neq i} R_{d_j-e}(g_j) \right) = 0, \forall i$.

Definition 2.2. For $f \in \mathcal{R}_d$, we say f **splits regularly** $n-1$ times if f is a sum of n non-zero forms of degree d in independent variables.[3]

It is no hard to show this is equivalent to the naive definition. However this is not true for $k[x_1, \dots, x_r]$ when $\operatorname{char} k > 0$.

We have the following basic observation

Lemma 2.3. *Let $f = g_1 + \cdots + g_n$ be a regular splitting of $f \in \mathcal{R}_d$, then for $e < d$ we have*

$$\operatorname{Ann}_R(f)_e = \bigcap_i \operatorname{Ann}_R(g_i)_e.$$

This implies "dummy" variables does not influence regular splittings:

Corollary 2.4. *If $d \ge 2$, $f \in \mathcal{R}_d$, then every regular splitting of f takes place inside $k[R_{d-1}(f)]^{DP} \subset \mathcal{R}$.*

Our main technique comes from the following two lemmas:

Lemma 2.5. *Given $f, g \in \mathcal{R}_d$, TFAE:*

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[2] Although I do not know, it is well-known that $R/\operatorname{Ann}_R(f)$ is Gorenstein of dimension 0, and any graded Artinian Gorenstein quotient arises this way.

[3] [Kle13] also discussed the notion of **degenerate splitting**, which is to say for some $f_t \in \mathcal{R}_d(t_1, \dots, t_m) = k(t_1, \dots, t_m)[x_1, \dots, x_r]_d^{DP}$, f_t splits regularly and $f_0 = f$.

a) $\mathrm{Ann}_R(f)_e \subset \mathrm{Ann}_R(g)_e, \forall e < d$*;*

b) $\mathrm{Ann}_R(f)_{d-1} \subset \mathrm{Ann}_R(g)_{d-1}, \forall e < d$*;*

c) $\partial g = A\partial f$ *for some* $A \in M_{r\times r}(k)$*;*

d) $R_1 \cdot \mathrm{Ann}_R(f)_{d-1} \subset \mathrm{Ann}_R(g)_d$*;*

e) $(\partial_1, \dots, \partial_r) \cdot \mathrm{Ann}_R(f) \subset \mathrm{Ann}_R(g)$*.*

Lemma 2.6. *For* $f \in \mathcal{R}_d, A \in M_{r\times r}(k)$*, TFAE:*

a) $\exists g \in \mathcal{R}_d$ *such that* $\partial g = A\partial f$*;*

b) $A\partial\partial^t(f)$ *is symmetric;*

c) $I_2([\partial A\partial]) \subset \mathrm{Ann})R(f)$*.*

Moreover, g in b) must be unique.

The following definition would play an important role.

Definition 2.7. For $f \in \mathcal{R}_d$, let M_f be the set of matrices

$$M_f := \{A \in M_{r\times r}(k) : I_2([\partial\ A\partial]) \subset \mathrm{Ann}_R(f)\}$$

where $\partial = (\partial_1, \dots, \partial_r)^t$, $[\partial\ A\partial]$ is a $r \times 2$ matrix and I_2 is the ideal generated by its 2×2 minors.

Definition 2.8. Let $\gamma_f : M_f \to \mathcal{R}_d$ be the map sending A to the unique $g \in \mathcal{R}_d$ such that $\partial g = A\partial f$, which is k-linear.

An important property is

Proposition 2.9. *If* $d \geq 3$*,* $f \in \mathcal{R}_d$*, then* M_f *is a k-algebra with all commutators belonging to* $\ker \gamma_f$*. Moreover* M_f *is commutative if* $\mathrm{Ann}_R(f)_1 = 0$*.*

Note that $\mathrm{Ann}_R(f)_1 = 0$ is equivalent to say f has no “dummy” variables. In the cases we care about we can always assume this condition holds.

How can we recover f from M_f?

Definition 2.10. Assume $M \subset M_{r\times r}(k)$.

a) Let
$$I(M) = \sum_{A\in M} I_2([\partial\ A\partial]), \widehat{I}(M) = \sum_{A,B\in M} I_2([A\partial\ B\partial]).$$

b) Let $X(M)$ be the graded R-module[4]

$$X(M) = \{f \in \mathcal{R} : M \subset M_f\}.$$

We list some properties

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[4] $X(M)$ is indeed an R-module since $\mathrm{Ann}_R(f) \subset \mathrm{Ann}_R(Df)$ for any $D \in R$.

Lemma 2.11. *a) If $I \in M$, then*[5] $I(M) \subset \widehat{I}(M) = I(M^2)$ *and* $I(M)_e = \widehat{I}(M)_e$ *for* $e \geq 3$.

b) If $I \in M$ *and* M *is closed under multiplication, then* $I(M) = \widehat{I}(M)$.

c) If $I \in M$, *let* M' *be the* k*-subalgebra of* $M_{r\times r}(k)$ *generated by* M, *then* $I(M') = \widehat{I}(M)$.

d) $M \subset M_f$ *iff* $I(M) \subset \mathrm{Ann}_R(f)$.

e) $X_d(M) = \{f \in \mathcal{R}_d : R_{d-2}(f) \subset X_2(M)\}$ *for* $d \geq 3$.

f) $I(M)^\perp = X_d(M)$ *for* $d \geq 0$.

g) $I(M) = \bigcap_{f\in X(M)} I(M_f) = \bigcap_{f\in X(M)} \mathrm{Ann}_R(f)$.

3 Idempotent and Splitting

Given a ring A, we say a subset of nonzero elements $\{e_1, \dots, e_n\}$ is a **coid** of A if the elements are complete orthogonal idempotents[6]. The length of such a coid is defined to be $\ell(\mathcal{E}) = n$. It is well-known that coids are in one-to-one correspondence with factorization into product of rings of A.

For two coids $\mathcal{E} = \{e_1, \dots, e_n\}, \mathcal{E}' = \{e'_1, \dots, e'_m\}$, we define

$$\mathcal{E} \otimes \mathcal{E}' = \{e_i e'_j : e_i e'_j \neq 0\}$$

which is again a coid. We list following basic facts.

Lemma 3.1. *a)* $\ell(\mathcal{E} \otimes \mathcal{E}') \geq \ell(\mathcal{E})$, *and the equality holds iff* $\mathcal{E} \otimes \mathcal{E}' = \mathcal{E}$. *Moreover, if the equality holds, then* $\mathcal{E}$ *refines* $\mathcal{E}'$ *in the sense that there exists a partition* $\{\mathcal{J}_1, \dots, \mathcal{J}_m\}$ *of* $\{1, \dots, n\}$ *such that* $e'_j = \sum_{i\in\mathcal{J}_j} e_i$.

b) If A *is Noetherian, then* A *contains a unique maximal coid* $\mathcal{E} = \{e_1, \dots, e_n\}$. *In particular, all of the idempotents in* A *are* $\sum_{i\in I} e_i$ *for* $\varnothing \neq I \subset \{1, \dots, n\}$.

c) If A *is further Artinian,* $\mathcal{E} = \{e_1, \dots, e_n\}$ *being the unique maximal coid, then each* $A_i = e_i A$ *is local Artinian with maximal ideal being* $\mathrm{nil}(A_i)$. *Moreover,* A *has exactly* n *prime ideals.*

For matrix algebra we will use, we have

Lemma 3.2. *Let* $M \subset M_{r\times r}(k)$ *be a commutative subalgebra.*

a) M *has a unique maximal coid* $\{E_1, \dots, E_n\}$.

b) $M_i = E_i M$ *is local Artinian with unique prime ideal* $M_i^{\mathrm{nil}} = \mathrm{nil}(M_i)$.

c) $k^r = \bigoplus_i \mathrm{im} E_i$.

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[5] Here M^2 denotes the set of all matrix products in M.

[6] i.e. $e_i^2 = e_i, e_i e_j = 0$ for $i \neq j$ and $\sum_i e_i = 1$.

d) Let[7] $\mathcal{I} = \{i : M_i = kE_i \oplus M_i^{\mathrm{nil}}$. *Then there are exactly* $|\mathcal{I}|$ *k-algebra homomorphisms* $M \to k$. *More explicitly, they are* $\lambda_i : M \to k, (cE_i, B) \mapsto c$ *for* $i \in \mathcal{I}$, *where* $B \in M_i^{\mathrm{nil}} \oplus \left(\bigoplus_{j\neq i} M_j\right)$. *Moreover* $V_{\lambda_i} = \mathrm{im}E_i$.

e) $\mathcal{I} = \{1, \ldots, n\}$ *iff k contains every eigenvalue of each element of M.*

Then we are ready to give our main result, which states that coids in M_f determine all of regular splittings of f. More precisely,

Theorem 3.3. *If* $d \geq 2$ *and* $\mathrm{Ann}_R(f)_1 = 0$, *let* $\mathrm{Coid}(M_f)$ *be the set of all coids* $\{e_1, \ldots, e_n\}$ *of orthogonal idempotents in* M_f, *and let*

$$\mathrm{Reg}(f) = \{\{g_1, \ldots, g_n\} : f = g_1 + \cdots + g_n \text{ is a regular splitting of } f\}.$$

Then the map $\{E_i\}_i \mapsto \{g_i = \gamma_f(E_i)\}_i$ *defines a bijection* $\mathrm{Coid}(M_f) \xrightarrow{\sim} \mathrm{Reg}(f)$. *Moreover, when* $d \geq 3$ *there is a unique maximal regular splitting.*

Proof. "Coid $\Longrightarrow$ Regular Splitting": Starting from some coid $\{E_1, \ldots, E_n\}$, let $g_i = \gamma_f(E_i)$. Fistly $\mathrm{Ann}_R(f)_1 = 0$ gives $R_{d-1}(f) = \mathcal{R}_1$, then $\partial R_{d-1}(f) = k^r$.

If $\partial g_i = E_i \partial f$, then $\partial R_{d-1}(g_i) = E_i \partial R_{d-1}(f) = \mathrm{im}E_i$, therefore[8] $R_{d-1}(g_i) = (\mathrm{im}E_i)^t x \subset \mathcal{R}_1$.

Note that $g_i \neq 0$ since $E_i \neq 0$ and $\mathrm{Ann}_R(f)_1 = 0$. Moreover

$$R_{d-1}(g_i) \cap \left(\sum_{j\neq i} R_{d-1}(g_j)\right) = \left(\mathrm{im}E_i \cap \left(\sum_{j\neq i} \mathrm{im}E_j\right)\right)^t x = 0$$

since E_i's form a coid. Therefore $f = \sum_i g_i$ is a regular splitting.

"Regular Splitting $\Longrightarrow$ Coid": There are unique $E_i \in M_f$ such that $\partial g_i = E_i \partial f$. We have $\gamma_f(E_i) = g_i$. $f = \sum_i g_i$ implies $\sum_i E_i = I$.

We have similarly from g_i's are in independent variables that $\mathrm{E_i} \cap \left(\sum_{j\neq i} \mathrm{im}E_j\right) = 0$.

Since for $i \neq j$ we have

$$E_i E_j = E_j(I - E_j) - \sum_{k\neq i,j} E_k E_j,$$

taking the image we obtain that $E_i E_j = 0$. This shows $\{E_1, \ldots, E_n\}$ is a coid of M_f.

When $d \geq 3$, M_f is a commutative k-algebra, and therefore has a unique maximal coid. This implies the uniqueness of maximal regular splitting. □

Remark 3.4. I am not sure when $d \geq 3$, the uniqueness of "regular splitting" holds for positive characteristic field, which seems to be true. Also I am not sure about how are $\mathcal{R}_d$ and $k[x_1, \ldots, x_r]_d$ related as representations of $\mathrm{GL}_r(k)$ if $\mathrm{char}\, k > 0$.

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[7] This actually depends on whether the residue field M_i/M_i^{nil}, as a finite extension of k, equals k.

[8] Here $x = [x_1, \ldots, x_r]^t$.

4 Some Results ans Questions I Care About

Let us together with our notions back to our dearest $\mathbb{C}[x_1,\dots,x_{n+2}]$. We denote $P_{n+2,d} = \mathbb{C}[x_1 \dots, x_{n+2}]_d$ and its subset $P^{\mathrm{sm}}_{n+2,d}$ of homogeneous polynomials of degree d that defines a smooth projective hypersurface (of dimension n) in $\mathbb{P}^{n+1}$. Note that when we also say a homogeneous polynomial F of one or two variables is smooth if $\partial_i F = 0, \forall i$ has no nontrivial solution.

Bases on Theorem 3.3, we first show a result on symmetric polynomials.

Lemma 4.1. *Let $V \subset \mathbb{C}^n$ be a nontrivial subspace such that if we assume the orbit of V under the S_n-action on $\mathbb{C}^n$ by permutations is $\{V_1 = V, \dots, V_k\}$, then $V_i \cap \left(\sum_{j\neq i} V_j\right) = 0$. Then up to S_n-action, V must be one of*

i) $\operatorname{span}\{(1,0,\dots,0)\}$*;*

ii) $\operatorname{span}\{1,\dots,1,x\}$ *for* $x \neq 1-n$*;*

iii) $\{\sum_i x_i = 0\}$*;*

iv) $\operatorname{span}\{(1,\omega,\omega^2)\}$ *if* $n=3$*;*

v) $\operatorname{span}\{(1,1,-1,-1)\}$ *if* $n=4$.

Proof. The cases $n = 1, 2$ are trivial. For $n \geq 3$ we need following facts:

Claim 4.2. *Assume $n \geq 3$, then*

a) $\mathbb{C}^n = V_{\mathrm{triv}} \oplus V_{\mathrm{std}}$ *is the irreducible decomposition as S_n-representation.*

b) $\mathbb{C}^n = V_{\mathrm{triv}} \oplus V_{\mathrm{std}}$ *is the irreducible decomposition as A_n-representation, except for $n = 3$ we have* $\mathbb{C}^3 = V_{\mathrm{triv}} \oplus \operatorname{span}\{(1,\omega,\omega^2)\} \oplus \operatorname{span}\{(1,\omega^2,\omega)\}$.

c) The only proper subgroups of S_n of index$\geq n$ are

  i) A_n, S_{n-1} *embedded in the natural way;*

  ii) D_8 *as the symmetric group of* 4 *vertices of a square if* $n = 4$*;*

  iii) S_5 *conjugate to* $\langle(1234),(3456)\rangle$ *if* $n = 6$*. There are six elements in the conjugate class.*

Denote $d = \dim V$ and $W = \sum_i V_i$. Then W is a S_n-invariant subspace of $\mathbb{C}^n$ of dimension kd. Thus $kd = 1, n-1, n$. If $k = 1$, then V is S_n invariant, which can be easily classified.

Since $k \leq n$ equals the index of stabilizer of V in S_n, V is the invariant subspace of those subgroups. Denote the stabilizer by S.

- $S = A_n$, then up to S_n action V is $\operatorname{span}\{(1,\dots,1)\}$, $\{\sum_i x_i = 0\}$ or $\operatorname{span}\{1,\omega,\omega^2\}$ if $n = 3$;
- $S = S_{n-1}$ as stabilizer of one element. Then up to S_n action V is $\operatorname{span}\{(1,\dots,1,x)\}$ for $x \neq 1-n$.
- $S = D_8$ and $n = 4$, then $k = 3$ and d must be 1. One can directly calculate, up to S_n action V can only be $\operatorname{span}\{(1,1,-1,-1)\}$.

- $S = S_5$ conjugate to $\langle(1234),(3456)\rangle$ and $n = 6$. Similarly $d = 1$. Since $C_6 = V_{\text{triv}} \oplus V_{\chi_6}$ is the irreducible representation of those S_5's, V can still only be $\operatorname{span}\{(1,\dots,1)\}$.

$\square$

Proposition 4.3. *Assume $n, d \geq 3$ and $F \in P_{n,d}$ is symmetric i.e. invariant under S_n-action. Let $m = \dim R_{d-1}(F)$ be the "necessary number of variables defining F". Then m together with the maximal regular splitting of F is one of*

i) *$m = n$ and F is unpartitionable;*

ii) *$m = 1$ and $F \sim \left(\sum_i x_i\right)^d$;*

iii) *$m = n$ and $F \sim \sum_i x_i^d$, or $F \sim \sum_i (s + cx_i)^d$ where $s = \sum_i x_i$ and $c \neq -n$;*

iv) *$m = n - 1$ and $F = G(x_1 - x_2, \dots, x_{n-1} - x_n)$ not belonging to cases v),vi) below is unpartitionable[9];*

v) *$m = n$ and $F \sim G(x_1 - x_2, \dots, x_{n-1} - x_n) + s^d$ where G does not belonging to cases v),vi).*

vi) *$m = 2, n = 3, 3|d$ and $F \sim (x_1 + \omega x_2 + \omega^2 x_3)^d + (x_1 + \omega^2 x_2 + \omega x_3)^d$.*

vii) *$m = 3, n = 4, 2|d$ and $F \sim (x_1 + x_2 - x_3 - x_4)^d + (x_1 - x_2 + x_3 - x_4)^d + (x_1 - x_2 - x_3 + x_4)^d$.*

viii) *$m = 3, n = 3, 3|d$ and*

$$F \in \operatorname{span}\{(x_1 + \omega x_2 + \omega^2 x_3)^d + (x_1 + \omega^2 x_2 + \omega x_3)^d, (x_1 + x_2 + x_3)^d\};$$

ix) *$m = 4, n = 4, 2|d$ and*

$$F \in \operatorname{span}\{(x_1 + x_2 - x_3 - x_4)^d + (x_1 - x_2 + x_3 - x_4)^d + (x_1 - x_2 - x_3 + x_4)^d, (x_1 + x_2 + x_3 + x_4)^d\}.$$

Moreover, if $F \in P^{\text{sm}}_{n,d}$ then it must be cases i),iii),v),viii) or ix). F in cases iii),viii) and ix) is linear equivalent to the Fermat polynomial.

Proof. Assume the maximal regular splitting of F is $F_1 + \cdots + F_k$. The uniqueness of maximal regular splitting gives a S_n-action on $\{F_i\}_i$. Apply Lemma 4.1 to $R_{d-1}(F_i)$ directly, we know what can a single orbit of the S_n-action on $\{F_i\}_i$ be. If there is exactly one orbit we get cases except v),viii) and ix). Moreover there are at most two orbits, and the cases with two orbits are v),viii) and ix). $\square$

Now we consider automorphism groups. Assume $n \geq 1, d \geq 3$ and $F \in P^{\text{sm}}_{n+2,d}$. Denote the smooth hyper surface defined by $F = 0$ in $\mathbb{P}^{n+1}$ by X_F.

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[9] The ring of translation-invariant symmetric polynomials is isomorphic to $\mathbb{C}[w_2, \dots, w_n]$ where $w_k = \sum_i (x_i - x_{\text{avg}})^k$, $x_{\text{avg}} = \frac{\sum_i x_i}{n}$ and w_k's are algebraically independent, see [Lip10, Corollary 2.3]. On the other hand, non-zero homogeneous elements of degree $d \geq 3$ in this ring must belong two this case or cases vi),vii) below.

Definition 4.4. Denote G_F by the subgroup of $\mathrm{GL}_{n+2}(\mathbb{C})$ preserving F, and PG_F the quotient of G_F by scalar. For $(n,d) \neq (1,3),(2,4)$, any embedding $X_F \hookrightarrow \mathbb{P}^{n+1}$ is linearly equivalent to the one given by F, then we can define $\mathrm{Aut}^l(X_F) < \mathrm{PGL}_{n+2}(\mathbb{C})$ the group of linear automorphisms of X, which equals PG_F.

Note that notions G_F and PG_F can be defined for elements in $P^{\mathrm{sm}}_{1,d}$ and $P^{\mathrm{sm}}_{2,d}$ as well.

Recall the classical result

Theorem 4.5 ([MM63]). *Assume $n \geq 1, d \geq 3$ and $F \in P^{\mathrm{sm}}_{n+2,d}$.*

a) $|G_F| < \infty$.

b) If $(n,d) \neq (1,3)$ or $(2,4)$, then $\mathrm{Aut}(X_F) = \mathrm{Aut}^l(X_F)$.

c) For generic F, PG_F is trivial.

For any $F \in P^{\mathrm{sm}}_{n+2,d}$, up to linear transformation we can write

$$F = x_1^d + \cdots + x_m^d + G(x_{m+1}, \ldots, x_{n+2})$$

where $G \in P^{\mathrm{sm}}_{n+2-m,d}$ is "purely non-Fermat". Then $G_F = ((C_d)^m \rtimes S_m) \times G_G$.

Alternatively,

$$F = x_{0,1}^d + \cdots + x_{0,n_0}^d + \sum_{j=1}^{k} F_j(x_{j,1}, \ldots, x_{n_j})$$

where $n_0 + \cdots + n_k = n$ and each $F_j \in P^{\mathrm{sm}}_{n_j,d}$ being unpartitionable.

Alternatively,

$$F = \bigoplus_{m \geq 1} \bigoplus_{H \in \overline{P^{\mathrm{sm,idcp}}_{m,d}}} H^{\oplus n_H}.$$

where $\overline{P^{\mathrm{sm,idcp}}_{m,d}}$ is the set of isomorphism classes under linear equivalence of unpartitionable polynomials in $P_{m,d}$, and $\sum_{n \geq 1} m \sum_{H \in \overline{P^{\mathrm{sm,idcp}}_{m,d}}} n_H = n$. Then

$$G_F = \prod_{m \geq 1} \prod_{H \in \overline{P^{\mathrm{sm,idcp}}_{m,d}}} (G_H^{n_H} \rtimes S_{n_H}).$$

Note that in general $|PG_F| = \frac{|G_F|}{d}$.

Both [YYZ25] and [I forgot] showed the following result

Theorem 4.6. *Assume $(n,d) \neq (1,3),(2,4)$, then the X achieving the maximal order of $|\mathrm{Aut}(X)|$ among all smooth hypersurfaces of degree d and dimension n is unique up to linear isomorphism. More explicitly, the corresponding X's are defined by*

The above theorem determines for $n \geq 1, d \geq 3$ the points achieving maximal values of function $|PG_F|$ on $\overline{P^{\mathrm{sm}}_{n,d}}$, except following cases[10]

- $n = 1$, $PG_F = \{*\}$.

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[10] Here one may confuse n and $n+2$.

(n,d)	$\mathrm{Aut}(X)$	$\lvert\mathrm{Aut}(X)\rvert$	X is isomorphic to X_F
$(1,4)$	$\mathrm{PSL}(2,7)$	168	$F = x_1^3x_2 + x_2^3x_3 + x_3^3x_1$
$(1,6)$	A_6	360	$F = 10x_1^3x_2^3 + 9(x_1^5 + x_2^5)x_3 - 45x_1^2x_2^2x_3^2 - 135x_1x_2x_3^4 + 27x_3^6$
$(2,6)$	$C_6.(S_4^2 \rtimes C_2)$	6912	$F = (x_1^5x_2 + x_2^5x_1)^{\oplus 2}$
$(2,12)$	$C_{12}.(A_5^2 \rtimes C_2)$	86400	$F = (x_1^{11}x_2 + 11x_1^6x_2^6 - x_1x_2^{11})^{\oplus 2}$
$(4,6)$	$\mathrm{PSU}(4,3).C_2$	6531840	$F = \sum\limits_{1\leq i\leq 6} x_i^6 + \sum\limits_{1\leq i\neq j\leq 6} 15x_i^4x_j^2 - \sum\limits_{1\leq i<j<k\leq 6} 30x_i^2x_j^2x_k^2 + 240\sqrt{-3}x_1x_2x_3x_4x_5x_6$
$(4,12)$	$C_{12}^2.(A_5^3 \rtimes S_3)$	186624000	$F = (x_1^{11}x_2 + 11x_1^6x_2^6 - x_1x_2^{11})^{\oplus 3}$
otherwise	$(C_d^{n+1}) \rtimes S_{n+2}$	$d^{n+1}(n+2)!$	$F = \sum_{i=1}^{n+2} x_i^d$

- $n = 2$, using the classification of finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$, the points achieving maximal $|PG_F|$ are
  - i) $F = x_1(x_1^3 - x_2^3)$ with $PG_F \simeq A_4$ of order 12, if $d = 4$;
  - ii) $F = x_1^5x_2 + x_2^5x_1$ with $PG_F \simeq S_4$ of order 24, if $d = 6$;
  - iii) $F = x_1^8 + 14x_1^4x_2^4 + x_2^8$ with $PG_F \simeq S_4$ of order 24, if $d = 8$;
  - iv) $F = x_1^{11}x_2 + 11x_1^6x_2^6 - x_1x_2^{11}$ with $PG_F \simeq A_5$ of order 120, if $d = 12$;
  - v) $F = x_1^{20} - 228x_1^{15}x_2^5 + 494x_1^{10}x_2^{10} + 228x_1^5x_2^{15} + x_2^{20}$ with $PG_F \simeq A_5$ of order 120, if $d = 20$;
  - vi) $F = x_1^{30} + 522x_1^{25}x_2^5 - 10005x_1^{20}x_2^{10} - 10005x_1^{10}x_2^{20} - 522x_1^5x_2^{25} + x_2^{30}$ with $PG_F \simeq A_5$ of order 120, if $d = 30$;
  - vii) $F = x_1^d + x_2^d$ with $PG_F \simeq D_{2d}$ of order $2d$, otherwise.
- $(n,d) = (3,3)$, from [YYZ24, Lemma 3.12] the maximal is achieved by the Fermat polynomial $F = x_1^3 + x_2^3 + x_3^3$ with $PG_F \simeq (C_3)^2 \rtimes S_3$ of order 54.
- $(n,d) = (4,4)$, by[11] [YYZ25, Theorem 6.1], the maximal is achieved by $F = x_1^4 + x_2^4 + x_3^4 + x_4^4 + 12x_1x_2x_3x_4$ with $PG_F \simeq C_2^4 \rtimes S_5$ of order 1920.

It is natural to ask for given (n,d), what is F with the second, third, e.t.c. maximal automorphism group? I think, as d is fixed and $n \to \infty$, the leading part of the order would be the Fermat part, so the answer depends completely on the following question:

Question 4.7. Assume $n \geq 1, d \geq 3$. When is the maximal value of $|PG_F|$ achieved in $\overline{P_{m,d}^{\mathrm{sm,nF}}}$?

And naturally we may ask

Question 4.8. Assume $n \geq 1, d \geq 3$. When is the maximal value of $|PG_F|$ achieved in $\overline{P_{m,d}^{\mathrm{sm,idcp}}}$?

Let us list known cases:

- $n = 1$, trivial.

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[11] The notion of primitivity of $\mathrm{Aut}(F)$ in [YYZ25], using our main theorem, is equivalent to the multiplicity $n_H \leq 1$ for any $H \in \overline{P_{m,d}^{\mathrm{sm,idcp}}}$. If F is partitionable, using previous results we see the automophism group is strictly smaller.

- $n=2$, the notion of purely non-Fermat and unpartitionable concides. Then PG_F is trivial except for special F's when $d=4,6,8,12,20,30$, and the maximal points are the same as listed above.

- $n=3$, the notion of purely non-Fermat and unpartitionable concides as well. Known for $d=3,4,6$.

- $(n,d)=(4,4)$, the maximal point stated above is unpartitionable.

- $(n,d)=(4,6),(4,12),(6,12)$, the maximal point stated are purely non-Fermat but not unpartitionable.

- $(n,d)=(6,6)$, the maximal point stated above is unpartionable by Proposition 4.3.

- $(n,d)=(4,3)$, by [Hos97, Theorem 5.3], the maximal is achievd by $x_1+x_2+x_3+x_4+x_5=x_1^3+x_2^3+x_3^3+x_4^3+x_5^3=0$ with automorphism group S_5 of order 120, which is unpartionable by Proposition 4.3.

- $(n,d)=(5,3)$, the maximal [12] is achieved by $F=x_1^2x_2+x_2^2x_3+x_3^2x_4+x_4^2x_5+x_5^2x_1$ with $|PG_F|\simeq \mathrm{PSL}(2,11)$ of order 660 (see [WY19, Example 3.1(5)]), which is unpartitionable by direct calculation on possible partitions.

- $(n,d)=(6,3)$, by our partial classification of the automorphism groups of smooth cubic fourfolds, maximal is achievd by $x_1+x_2+x_3+x_4+x_5+x_6+x_7=x_1^3+x_2^3+x_3^3+x_4^3+x_5^3+x_6^3+x_7^3=0$ with automorphism group S_7 of order 5040, which is unpartionable by Proposition 4.3.

I am not sure about the (asymptotic) amount of maximal orders in those two questions.

Denote the maximal value of $|PG_F|$ on $\overline{P_{n,d}^{\mathrm{sm,nF}}}$ and $\overline{P_{n,d}^{\mathrm{sm,idcp}}}$ by $M_{n,d}^{\mathrm{nF}}$ and $M_{n,d}^{\mathrm{idcp}}$ respectively.

We take a quotient and put a total order on $\overline{P_{n,d}^{\mathrm{sm}}}(\overline{P_{n,d}^{\mathrm{sm,nF}}}$ *resp.*) by comparing the order of PG_F. We denote the quotient by $\mathcal{P}_{n,d}(\mathcal{P}_{n,d}^{\mathrm{nF}}$ *resp.*), and elements of $\mathcal{P}_{n,d}(\mathcal{P}_{n,d}^{\mathrm{nF}}$ *resp.*) are

$$F_{n,d}^1>\cdots>F_{n,d}^{N_{n,d}}$$

$$(F_{n,d}^{\mathrm{nF},1}>\cdots>F_{n,d}^{\mathrm{nF},N_{n,d}^{\mathrm{nF}}}\ \textit{resp.}).$$

Denote the Fermat polynomial of n variables and degree d by $\mathrm{Ferm}_{n,d}$

If the answers are asymptotically not too large, we would have following result

Proposition 4.9. *If for any fixed k and sufficiently large n, $M_{n,d}^{\mathrm{nF}}<d^{n-k}(n-k)!$, then for fixed d and N, when n is sufficiently large, then the first to the N'th largest elements in $\mathcal{P}_{n,d}$ are exactly*

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[12] To check maximality, adding x_6^3 term and check our partial classification of the automorphism groups of smooth cubic fourfolds. There are no groups of strictly larger and splits out exactly one Fermat term.

the first to the N'th terms in the following:

$$\begin{aligned}
F^1_{n,d} &= F^{\mathrm{nF},1}_{0,d} \oplus \mathrm{Ferm}_{n,d} = \{\mathrm{Ferm}_{n,d}\},\\
F^2_{n,d} &= F^{\mathrm{nF},1}_{2,d} \oplus \mathrm{Ferm}_{n-2,d}, \dots, F^{1+N^{\mathrm{nF}}_{2,d}}_{n,d} = F^{\mathrm{nF},N^{\mathrm{nF}}_{2,d}}_{2,d} \oplus \mathrm{Ferm}_{n-2,d},\\
F^{2+N^{\mathrm{nF}}_{2,d}}_{n,d} &= F^{\mathrm{nF},1}_{3,d} \oplus \mathrm{Ferm}_{n-3,d}, \dots, F^{1+N^{\mathrm{nF}}_{2,d}+N^{\mathrm{nF}}_{3,d}}_{n,d} = F^{\mathrm{nF},N^{\mathrm{nF}}_{3,d}}_{3,d} \oplus \mathrm{Ferm}_{n-3,d},\\
&\dots\\
F^{2+N^{\mathrm{nF}}_{2,d}+\cdots+N^{\mathrm{nF}}_{k-1,d}}_{n,d} &= F^{\mathrm{nF},1}_{k,d} \oplus \mathrm{Ferm}_{n-k,d}, \dots, F^{1+N^{\mathrm{nF}}_{2,d}+\cdots+N^{\mathrm{nF}}_{k,d}}_{n,d} = F^{\mathrm{nF},N^{\mathrm{nF}}_{k,d}}_{k,d} \oplus \mathrm{Ferm}_{n-k,d},
\end{aligned}$$

where k is the minimal such that $1 + N^{\mathrm{nF}}_{2,d} + \cdots + N^{\mathrm{nF}}_{k,d} \geq N$.

Proof. We only have to show

$$M^{\mathrm{nF}}_{n',d} d^{n-n'} (n-n')! < d^{n-k}(n-k)!$$

for $k+1 \leq n' \leq n$. The above is equivalent to

$$M^{\mathrm{nF}}_{n',d} < d^{n'-k} \frac{(n-k)!}{(n-n')!}.$$

If n' satisfies $M^{\mathrm{nF}}_{n',d} < d^{n'-k}(n'-k)!$, then

$$M^{\mathrm{nF}}_{n',d} < d^{n'-k}(n'-k)! \leq d^{n'-k} \frac{(n-k)!}{(n-n')!}.$$

For small n' just let n be sufficiently large. □

The following proposition from [Col08] may be useful (but not enough?)

Proposition 4.10 (Part of [Col08, Theorem A])**.** *If $n > 12$, G is a primitive subgroup of $GL_n(\mathbb{C})$, then $[G : Z(G)] \leq (n+1)!$.*

The following conjectures is not so responsible, since the above proposition requires primitivity.

Conjecture 4.11. *For large n,*

a) $M^{\mathrm{nF}}_{n,d} \leq n! \left(\frac{d}{1+\epsilon}\right)^n$, and

b) For $F \in P^{\mathrm{sm,nF}}_{n,d}$, $|Z(G_F)| \leq \left(\frac{d}{1+\epsilon}\right)^n$.

[May13] also lead us to consider maximal abelian groups. In total we may ask

Question 4.12. Given n, d, what is the maximal value of $|PG_F|$, $|Z(PG_F)|$ and the maximal order of abelian subgroup of PG_F, for $F \in P^{\mathrm{sm}}_{n,d}, P^{\mathrm{sm,nF}}_{n,d}$ or $P^{\mathrm{sm,idcp}}_{n,d}$?

Note that [Sza96] showed

Theorem 4.13. *For $n \geq 3$, $F \in P_{n,d}^{\mathrm{sm}}$ and $G \subset PG_F$, we have*

$$|G| \leq \begin{cases} d^{n-1}, & \textit{if } G \textit{ is abelian} \\ d^{(n-1)(2n-3)}, & \textit{if } G \textit{ is nilpotent} \\ d^{\frac{(n-1)n(2n-1)}{2}}. & \textit{if } G \textit{ is solvable} \end{cases}$$

Anyway the second and the third bound is loose.

5 Positive Characteristic Case

In this section we assume k is a algebraically closed field of characteristic $p > 0$. Anyway it will turn out that it is equivalent to consider just $\overline{\mathbb{F}_p}$.

One can also ask questions of uniqueness of addictive decomposition, and when is maximal size of automorphism group of hypersurfaces achieved. The first question is not known for me yet, and I will update the note once I become sure about it. Here I give some basic facts.

If one wants to follow methods in [YYZ25] to deal with positive characteristic case, there are truly analogues for Jordan and Collin's theorems like results in [Col07] and [LP11] turn to be extremely complicated and not usable in our problem. Those results must be established upon imaginable objects like Lie-type groups in characteristic p.

I think, one can show the following baby result, which is not so hard to prove and may be known earlier. Anyway we give a proof here.

Proposition 5.1. *Fix $n \geq 2, d \geq 3$, when p is sufficiently large, possible automorphism groups of smooth (n, d)-hypersurfaces is the same as the characteristic 0 case. More explicitly*[13]*, if $p \geq (n-1)d + 2$, then possible automorphism groups of smooth (n, d)-polynomials over k is just ones of smooth (n, d)-polynomials over $\mathbb{C}$ with order not divisible by p, in the sense of representations*[14].

We only[15] have to show

Proposition 5.2. *Fix $n \geq 2, d \geq 3$. If $p \geq (n-1)d + 2$, then any $G \in \mathrm{GL}(n, k)$ of order p cannot preserve any smooth polynomial $F \in k[x_1, \ldots, x_n]_d$.*

To show this we have to use the theory of modular representations. The structure of ring of invariant polynomials would be complicated in the modular case. Anyway we still have some descriptions. We will use results in [Weh11].

If $p \geq n$, any $A \in \mathrm{GL}(n, k)$ of order p is conjugate to a direct sum of Jordan blocks with eigenvalues 1, say $\begin{bmatrix} J_{n_1} & & \\ & \ddots & \\ & & J_{n_k} \end{bmatrix}$ where $n_1 \leq \cdots \leq n_k$ and J_m is the $m \times m$ Jordan block with eigenvalue 1. The C_p action is generated by $F(x_1, \ldots, x_n) \mapsto F((x_1, \ldots, x_n)A)$.

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[13] At the very beginning, I think this bound could be $\max\{n, d+1\}$. I am not sure whether this bound is OK, and just give the proof for $(n-1)d + 2$.

[14] It is well known that when $p \nmid |G|$, representation theories of G over $\mathbb{C}$ and $\overline{\mathbb{F}_p}$ are the same. To give the correspondence one need to pick choose a number field like $K = \mathbb{Q}[\zeta_{|G|}]$ and define the representations by $\mathcal{O}_K[G]$.

[15] Following the above footnote, we can consider polynomials defined over $\mathcal{O}_K$. As smoothness is equivalent to non-existence of non-trivial solution of a set of homogeneous equations, which is equivalent to the non-zero-ness of corresponding resultant ($\in \mathcal{O}_K$). For $p \notin |G|$, those non-zero-ness are equivalent in characteristic 0 and p.

Lemma 5.3. *If $p \geq (n-1)d+2$ and $A \in \mathrm{GL}(n,k)$ is of order p. Then any $F \in k[x_1,\dots,x_n]_d$ invariant under A lies in $\mathbb{Z}[x_1,\dots,x_n]_d^{\widetilde{A}} \otimes k$ where $\widetilde{A} = \begin{bmatrix} J_{n_1} & & \\ & \ddots & \\ & & J_{n_k} \end{bmatrix} \in \mathrm{GL}(n,\mathbb{Z})$ has the same Jordan blocks as A.*

Proof. We can assume $n_1 > 1$ and F is homogeneous for each set of variables corresponding to a Jordan block. By[16] [Weh11, Theorem 9.14] we only have to show the "transfer" part equals zero. That is equivalent to show the transfer induced by any monomial of degree$\leq d$ is zero.

Let[17] the set of variables be $\{x_{i,j} : 1 \leq i \leq k, 1 \leq j \leq n_i\}$. Pick a monomial $\prod_{i,j} x_{i,j}^{d_{i,j}}$ with $\sum_{i,j} d_{i,j} \leq d$. Then the induced transfer is

$$\begin{aligned}\mathrm{Tr}^{C_p}\left(\prod_{i,j} x_{i,j}^{d_{i,j}}\right) &= \sum_{s=0}^{p-1}\prod_{i,j}(((x_1,\dots,x_n)A^s)_{i,j})^{d_{i,j}} \\ &= \sum_{s=0}^{p-1}\prod_{\substack{1\leq i\leq k,\\ 1\leq j\leq n_i}}\left(\sum_{l=0}^{j-1}\binom{s}{l}x_{j-l}\right)^{d_{i,j}}.\end{aligned}$$

If we recognize s as a variable, since all of l's in the summation is not greater than $j-1 \leq n_i - 1 \leq n-1$, we have

$$\prod_{\substack{1\leq i\leq k,\\ 1\leq j\leq n_i}}\left(\sum_{l=0}^{j-1}\binom{s}{l}x_{j-l}\right)^{d_{i,j}} \in \mathbb{F}_p[x_1,\dots,x_n][s]$$

whose degree with respect to s is not larger than

$$\sum_{\substack{1\leq i\leq k,\\ 1\leq j\leq n_i}}(j-1)d_{i,j} \leq \sum_{i=1}^{k}(n_i-1)\sum_{j=1}^{n_i}d_{i,j} \leq (n-1)d \leq p-2$$

Therefore the summation $\sum_{r=0}^{p-1}$ is zero. □

Proof of Proposition 5.1. Let F be a smooth (n,d)-polynomial over k which is invariant under A.

Using Lemma 5.3 we deduce that $F \in \mathbb{Z}[x_1,\dots,x_n]_d^{\widetilde{A}} \otimes k$. Let

$$R = (\mathbb{F}_p[\{\text{coefficients of } F\}])_{r_0} \subset k$$

be a finite-type $\mathbb{F}_p$-algebra localized at r_0, where r_0 is the resultant of $\left(\frac{\partial F}{\partial x_i}\right)_i$. Then[18] there exists some q, a power of p, and a k-algebra homomorphism $R \to \mathbb{F}_q$. Applying this homomorphism to coefficients of F we deduce a A-invariant (n,d)-polynomial F_1 lying in $\mathbb{Z}[x_1,\dots,x_n]_d^{\widetilde{A}} \otimes \mathbb{F}_q$. From the resultant we see it is also smooth.

Pick[19] a number field K and some prime ideal $\mathfrak{p}$ such that $\mathcal{O}_K/\mathfrak{p} \simeq \mathbb{F}_p$. Choose some

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[16] The original theorem in [Weh11] is stated over $\mathbb{F}_p$, clearly its general version is enough for us.

[17] Here our notion is a bit different from that in [Weh11] since we use the A-action from the right side.

[18] Here we use the following fact: if K is an arbitrary field and $L = K(t_1,\dots,t_m)$ is a finitely-generated extension of K, then there exists a finite extension K'/K and a k-algebra homomorphism $K[t_1,\dots,t_m] \to K'$. This can be shown using the strong version of Noether's normalization theorem.

[19] For example, let $K = \mathbb{Q}[\zeta_{q-1}]$ and let $\mathfrak{p}$ be any prime ideal over p.

$F_2 \in \mathbb{Z}[x_1, \dots, x_n]_d^{\widetilde{A}} \otimes \mathcal{O}_K = \mathcal{O}_K[x_1, \dots, x_n]_d^{\widetilde{A}} \subset \mathbb{C}[x_1, \dots, x_n]_d^{\widetilde{A}}$ such that its image under the map $\mathcal{O}_K[x_1, \dots, x_n]_d^{\widetilde{A}} \to \mathbb{Z}[x_1, \dots, x_n]_d^{\widetilde{A}} \otimes \mathbb{F}_q$ is F_1. Also from the resultant F_2 is smooth.

However $\mathrm{Aut}(F_2) > \langle \widetilde{A} \rangle$ is infinite, which contradicts to F_2 is smooth by [MM63]. $\square$

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