Notes on Finiteness, Linearity and Generic Triviality of Automorphism Groups of Smooth Hypersurfaces

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Contents

The goal of this note is to study some basic results on automorphisms of smooth hypersurfaces. Our main references are [Cha78][1] and [MM63]. Our main results are mostly first proved in those papers.

Throughout the note we assume k is an algebraically-closed field, $n \geq 1, d \geq 3$ and $X = X_{n,d} \subset \mathbb{P}^{n+1}$ is a smooth hypersurface of dimension n and degree d. The groups of birational maps from X to itself, automorphisms of X and linear automorphisms[2] of X are denoted by $\mathrm{Bir}(X)$, $\mathrm{Aut}(X)$ and $\mathrm{Lin}(X)$ respectively. Note that $\mathrm{Lin}(X)$ would be an algebraic group.

Assume X is defined by $F(x_1, \dots, x_{n+2}) = 0$. Denote $F_i := \frac{\partial F}{\partial x_i}$.

Our main results are

Theorem 0.1. *Using the notations above, we have*

a) (Finiteness) $|\mathrm{Lin}(X)| < \infty$.

b) (Linearity) If $(n, d) \neq (1, 3), (2, 4)$, *then* $\mathrm{Aut}(X) = \mathrm{Lin}(X)$.

c) (Generically Triviality)[3] *If* $n \geq 2$, *then generic* X *has* $\mathrm{Aut}(X) = \{*\}$.

[1] About this paper, most papers write Hai Chau, Chang(張海潮) for the author, as well as one can check from the original paper. However JSTOR said there are two authors, H. C. Chang and 簡燦榮. Maybe there are some mistakes.

[2] i.e. induced by some element in $PGL_{n+2}(k) = \mathrm{Aut}(\mathbb{P}^{n+1})$ preserving X for some fixed embedding $X \hookrightarrow \mathbb{P}^{n+1}$.

[3] Actually as mentioned at its last page, [MM63] does not provide the proof of this result for $(n, d) = (2, 4)$. For general smooth projective curves C_g of genus $g \geq 2$, the automorphism group is always finite, and generic C_g has automotphism group $\simeq \mathbb{Z}/2\mathbb{Z}$ if $g = 2$, which is given by the hyperelliptic involution; $\simeq \{*\}$ if $g \neq 3$. We will not give the proof of this result as well.

1 Finiteness

Roughly speaking, the finiteness of linear part comes from algebraic relations, especially from smoothness. We could always consider the identical connected component of the algebraic group. If it is not a single point, new algebraic relations would be produced. Luckily, smoothness provide us strong algebraic properties for derivatives. Anyway we have to deal with problems produced by positive characteristics.

In one of our cases we need the following lemma:

Lemma 1.1. *Let k be an arbitrary field. Suppose $n, d \geq 2$ and $f_1, \ldots, f_{n+2} \in k[x_1, \ldots, x_{n+2}]$ are homogeneous of degree d. Let $\mathfrak{a} = (f_1, \ldots, f_{n+2})$. If $\operatorname{depth}(\mathfrak{a}) \leq 1$ and $\sum_i x_i f_i = 0$, then that is the only linear relation in $\{x_i f_j\}_{i,j}$.*

Proof. Firstly we have $\operatorname{depth}(\mathfrak{a}) = 1$. Otherwise since $k[x_1, \ldots, x_{n+2}]$ is Cohen-Macaulay of dimension $n+2$, $F_1, \ldots, F_{n+2}$ is a regular sequence. The vanishing of Koszul homology implies $x_1 \in (f_2, \ldots, f_{n+2})$, which leads a contradiction.

By passing to $\overline{k}$ we can assume k is infinite. Assume l_i's are linear, $\sum_i f_i l_i = 0$ and $(l_i)_i$ is not proportional to $(x_i)_i$. Through generic linear transform we may assume[4] $\operatorname{depth}((f_2, \ldots, f_{n+2})) = 1$. WLOG we assume x_2 occurs in l_1, then $x_1 l_2 - x_2 l_1 \neq 0$. Since

$$\sum_{i=2}^{n+2} f_i(x_1 l_i - x_i l_1) = 0,$$

also from depth condition we have $f_2, \ldots, f_{n+2}$ is a regular sequence, which implies $x_1 l_2 - x_2 l_1 \in (f_2, \ldots, f_{n+2})$. If $d > 2$ this leads a contradiction.

When $d = 2$, let $\varphi_i = x_1 l_i - x_i l_1$. Since φ_2 contains x_2^2 term but $\varphi_3, \ldots, \varphi_{n+2}$ do not, and φ_i contains $x_2 x_i$ term but φ_j's do not where $3 \leq i \leq n+2, j \notin \{1, 2, i\}$, $\varphi_2, \ldots, \varphi_{n+2}$ are linearly independent. Since they are all in $(f_2, \ldots, f_{n+2})$, we have $(f_2, \ldots, f_{n+2}) = (\varphi_2, \ldots, \varphi_{n+2}) \subset (x_1, l_1)$.

However $\operatorname{depth}((x_1, l_1)) \geq n + 2 - 2 = n$, which contradicts to $n \geq 2$. $\square$

Then we can prove the finiteness of linear automorphism group.

Theorem 1.2. $|\operatorname{Lin}(X)| < \infty$.

Proof. Since $\operatorname{Lin}(X)$ is an algebraic group, we only have to show $\operatorname{Lin}(X)^\circ$ is trivial. Since the projection is open, the inverse image of $\operatorname{Lin}(X)^\circ$ in $GL_{n+2}(k)$ is connected. We only have to show any connected subgroup G of $GL_{n+2}(k)$ containing the groups of scalars is exactly the groups of scalars.

i) **The case** $\operatorname{char}(k) \nmid d$**.** Let $\mathfrak{g}$ be the Lie algebra of G. We only have to show $\mathfrak{g} = kI \subset \mathfrak{gl}_{n+2}$.

For $g = (g_{ij}) \in G \subset GL_{n+2}(k)$, we have

$$F\left(\sum_j g_{1j} x_j, \ldots, \sum_j g_{(n+2)j} x_j\right) = \chi(g) F(x_1, \ldots, x_{n+2})$$

[4]That is somewhat nontrivial commutative algebra. If later I am sure about a correct proof or references I may complete the proof here. The main reason is that the height of $\mathfrak{a}$ is exactly $n+1$.

for some character χ of G. Applying any tangent vector $\xi = (\xi_{ij}) \in \mathfrak{g} \subset \mathfrak{gl}_{n+2}$ we deduce that

$$\sum_i F_i(x) \left(\sum_j \xi_{ij} x_j \right) = \langle \xi, \chi \rangle F(x).$$

Using Euler identity we have

$$\sum_i F_i(x) \left(\sum_j \xi_{ij} x_j - \frac{\langle \xi, \chi \rangle}{d} x_i \right) = 0.$$

Since F is smooth and $\mathrm{char}(k) \nmid d$, $(F, F_1, \ldots, F_{n+2}) = (F_1, \ldots, F_{n+2})$ has depth 0 as ideal of $k[x_1, \cdots, x_{n+2}]$. Since $k[x_1, \ldots, x_{n+2}]$ is Cohen-Macaulay of dimension $n+2$, $F_1, \ldots, F_{n+2}$ is a regular sequence. The vanishing of Koszul homology implies for each i that

$$\sum_j \xi_{ij} x_j - \frac{\langle \xi, \chi \rangle}{d} x_i \in (F_1, \ldots, F_{n+2}),$$

which further gives $\xi_{ij} = \frac{\langle \xi, \chi \rangle}{d} \delta_{ij}$.

ii) **The case** $\mathrm{char}(k)|d$**.** Since G is generated by any Borel subgroup and any Borel subgroups contains $k^\times I \simeq \mathbb{G}_\mathrm{m}$, replacing G by its Borel subgroup we can assume G is solvable. Then G is the semi-direct product of a maximal torus T and a connected unipotent group U. We only have to show $T = k^\times I$ and $U = \{I\}$.

Firstly consider torus T. Through linear transform we can assume all of elements of T are diagonal, say of thdiag$(\chi_1(t), \ldots, \chi_{n+2}(t))$ for $t \in T$. Then

$$F(\chi_1(t)x_1, \ldots, \chi_{n+2}(t)x_{n+2}) = \chi(t) F(x_1, \cdots, x_{n+2}).$$

Since F is smooth, there are some i_1 such that F contains $x_1^{d-1} x_{i_1}$ term, thus $\chi_1^{d-1}\chi_{i_1} = \chi$. Similarly there are i_k's such that $\chi_{i_k}^{d-1}\chi_{i_{k+1}} = \chi$. Consider the minimal k such that $i_{k+1} = 1$, then $\chi_1^{1-(1-d)^{k+1}} = \chi^{\sum_{l=0}^{k}(1-d)^l}$, which implies $\chi = \chi_1^d$. Similar arguments implies $\chi_1 = \cdots = \chi_{n+2}$.[5]

Then we have to treat unipotent group U. Since U has no nontrivial character, U actually preserves F. Through change of variables we can assume elements of U are of the form $\begin{bmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{bmatrix}$. Let $\mathfrak{u}$ be the Lie algebra of U. Take $\eta = (\eta_{ij}) \in \mathfrak{u} \subset \mathfrak{gl}_{n+2}$, then $\eta_{ij} = 0$ if $i \geq j$.

Similarly as in i), we obtain

$$\sum_{i=1}^{n+2} F_i(x) \sum_{j=i+1}^{n+2} \eta_{ij} x_j = 0.$$

Note that Euler's identity we have $\sum_i F_i(x) x_i = 0$. Applying Lemma 1.1 we deduce that $\eta = 0$, and the conclusion follows. □

[5]Here we used the fact that any homomorphism from a smooth connected algebraic group to μ_m is trivial.

2 Linearity

Then we come to the proof of linearity. To conclude linearity we have to use the theory of linear systems, especially Lefschetz-type theorems. The curve case is not so hard.

Lemma 2.1 ([Cha78, Lemma 1])**.** *Let X be a smooth plane curve of degree≥ 4. Then the hyperplane section is complete.*[6]

Proof. We only have to consider generic hyperplane(which is just a line) section divisor $H = [p_1] + \cdots + [p_d]$, which satisfies p_i's are distinct points on X. Using Riemann-Roch formula we deduce that

$$h^0(X, \mathscr{O}(H)) - h^0(X, \mathscr{O}(K_X - H)) = d - g(X) + 1 = -\frac{d(d-5)}{2}.$$

On the other hand, $H^0(X, K_X) = H^0(X, \mathscr{O}_X(d-3))$ is just the space of homogeneous polynomials of degree $d-3$. Since H is cut out by some line, $H^0(X, \mathscr{O}(K_X - H))$ is just the space of homogeneous polynomials of degree $d-3$ containing this line as a factor. Therefore $h^0(X, \mathscr{O}(K_X - H))$ is just the dimension of the space of homogeneous polynomials of degree $d-4$, which equals $\binom{d-2}{2}$.

Therefore $h^0(X, \mathscr{O}(H)) = 3$, and the lemma follows. □

Theorem 2.2 ([Cha78, Theorem 2])**.** *If $n = 1$, $d \geq 4$, then* $\mathrm{Aut}(X) = \mathrm{Lin}(X)$.

Proof. Take $\sigma \in \mathrm{Aut}(X)$ and generic line section divisor $H = [p_1] + \cdots + [p_d]$ where p_i's are distinct points on X. We just have to show $\sigma(H)$ is again a line section. Denote $q_i = \sigma(p_i)$.

Since $h^0(X, K_X) - h^0(X, \mathscr{O}(K_X - H)) = d - 2$, there are $d-2$ points among p_i's such that, for any homogeneous polynomial of degree $d-3$ corresponding to elements of $|K_X|$, one the polynomial vanishes at those $d-2$ points, it would vanish on H. This property again holds for q_i's. Assume that $d-2$ points are $q_1, \ldots, q_{d-2}$. Consider linear form l_{12} joining q_1 and q_2, and l_i passing through q_i but not through q_{d-1} or q_d for $3 \leq i \leq d-2$. Then $l_{12} \prod_i l_i$ is a degree-$(d-3)$ homogeneous polynomial vanishing on $q_1, \ldots, q_{d-2}$. Thus q_{d-1} and q_d lies on l_{12}. Similarly all of $q_1, \ldots, q_{d-2}$ lies in the line joining q_{d-1} and q_d, thus the theorem follows. □

For higher dimension cases, recall Lefschetz theorem for Picard groups, which is called Severi-Lefschetz-Andreotti in [MM63]:

Theorem 2.3 (Lefschetz theorem for Picard groups, Example 3.1.25 in [Laz17])**.** *Let Y be a smooth projective variety of dimension ≥ 4, and $D \subset Y$ a reduced effective ample divisor. Then the map* $\mathrm{Pic}(Y) \to \mathrm{Pic}(D)$ *is an isomorphism.*

Theorem 2.4. *If $n \geq 2$ and $(n, d) \neq (2, 4)$, then* $\mathrm{Aut}(X) = \mathrm{Lin}(X)$.

Proof. Let L_1 be the linear system of hyperplane sections on X.

If $n \geq 3$, using Lefschetz theorem for Picard groups, any effective divisor on X is cut out by some hyperplane in $\mathbb{P}^{n+1}$. Therefore L_1 is complete and gives a unique base of the additive semi-group of the linear equivalence classes of effective divisors on X. Since L_1 is invariant under $\mathrm{Aut}(X)$, elements in X must be linear.

[6] i.e. gives a complete linear system.

If $n=2$, L_1 is also complete since X is projectively normal. If $d=3$, L_1 is just $\mathscr{O}_X(1)=-K_X$, which is again invariant under $\mathrm{Aut}(X)$. If $d\geq 5$, then $K_X=\mathscr{O}_X(d-4)$ is invariant under $\mathrm{Aut}(X)$. If $\sigma\in\mathrm{Aut}(X)$ does not preserve L_1, there is some $D\in L_1$ not preserved by σ i.e. $\sigma(D)-D\not\sim 0$. However invariance of K_X implies $(d-4)(\sigma(D)-D)\sim 0$, thus $d\geq 6$. Therefore $(d-4)(\sigma(D)-D)=(\phi)$ for some $\phi\in K(X)^\times$. Then $\phi^{\frac{1}{d-4}}$ gives an unramified étale $(d-4)$-cover of X, which contradicts to $\pi_1^{\text{ét}}(X)=\{*\}$. □

3 Generic Hypersurfaces

Let us show that when $n\geq 2$, generic X has trivial linear automorphism group. Combining this with linearity, we deduce that the automorphism group is trivial if $(n,d)\neq(2,4)$. However as mentioned in the footnote, for general (n,d), generic hypersurfaces has trivial automorphism group, which we will not show in this note.

Theorem 3.1. *If $n\geq 2$, generic X has trivial linear automorphism group.*

Proof. Let us pass to $K:=\overline{k\left(\{t_r\}_{1\leq r\leq\binom{n+d+1}{d-1}}\right)}$. Assume F is a homogeneous polynomial of degree d in $K[x_1,\ldots,x_{n+2}]$ with coefficients algebraically independent over k. Assume $A=(a_{ij})_{1\leq i,j\leq n+2}\in\mathrm{GL}_{n+2}(K)$ satisfies $F(A(x))=cF(x)$ where $c\in K^\times$. We have to show A is actually a scalar matrix.

Write $A=A_sA_u=A_uA_s$ where A_s is semi-simple and A_u is unipotent. As matrices leaving F semi-invariant form a closed algebraic subgroup of $\mathrm{GL}_n(K)$, which must be closed under taking semi-simple part and unipotent part, we just have to consider the cases A itself being semi-simple or unipotent.

i) **The case** A is semi-simple. Then there are some $P\in\mathrm{GL}_{n+2}(K)$ such that

$$PAP^{-1}=B=\begin{bmatrix}\alpha_1 I_{n_1} & & & 0\\ & \alpha_2 I_{n_2} & & \\ & & \ddots & \\ 0 & & & \alpha_s I_{n_s}\end{bmatrix},$$

where α_i's are pair-wisely distinct and $\sum n_i=n+2$. Let $H\simeq\prod\mathrm{GL}_{n_i}(k)\subset G=\mathrm{GL}_{n+2}(k)$. Then $H(K)$ is the centralizer of B in $G(K)$. Consider variety G/H over k of dimension $2\sum_{i<j}n_in_j$, with projection map $G\xrightarrow{\pi}G/H$. Let $p\in G/H$ be the image of $P\in G(K)$. Consider $\pi^{-1}(p)=G\times_{G/H}\operatorname{Spec}\kappa(p)\subset T\cdot H(K)$, which is can be recognized as an variety over $\kappa(p)$. Take an algebraic[7] point $Q\in PH$. Then $QAQ^{-1}=B$. Moreover, we have

$$\operatorname{tr.deg}_k S\leq\operatorname{tr.deg}_k\kappa(p)\leq\dim_k(G/H)=2\sum_{i<j}n_in_j.$$

Let $G(x)=F(Q^{-1}(x))$, then $G(B(x))=cG(x)$. Since $F=G(Q(x))$ is generic, we only have to show[8] more than $2\sum_{i<j}n_in_j$ monomials are absent in G.

[7] i.e. with residue field algebraic over $\kappa(p)$.

[8] Therefore, $\operatorname{tr.deg}_k\{\text{coefficients of }F\}\leq\operatorname{tr.deg}_k\{\text{coefficients of }G\}+\operatorname{tr.deg}_k S<\binom{n+d+1}{d-1}$, which contradicts to F is generic.

Assume $s > 1$. Re-denote our variables by $x_j^{(i)}$ where $1 \leq j \leq n_i$. Denote $(x_1^{(i)}, \dots, x_{n_i}^{(i)})$ by $x^{(i)}$. Let $M = \left\{ \text{cubic monomials appear in } \frac{G}{x_1^{(1)^{d-3}}} \right\}$. For i, j, l pairwise distinct, consider set of monomials[9]

$$\begin{aligned} M_i &= \left\{ \text{terms cubic in } x^{(i)} \right\} \text{ of order } \frac{n_i(n_i+1)(n_i+2)}{6}, \\ M_{ij} &= \left\{ \text{terms quadratic in } x^{(i)} \text{ and linear in } x^{(j)} \right\} \text{ of order } \frac{n_j n_i(n_i+1)}{2}, \\ \text{and } M_{ijl} &= \left\{ \text{terms linear in all of } x^{(i)}, x^{(j)}, \text{ and } x^{(l)} \right\} \text{ of order } n_i n_j n_l. \end{aligned}$$

Note that for $i < j$, at most one of M_{ij} and M_{ji} can have nonempty intersection with M since $\alpha_i^2 \alpha_j \neq \alpha_j^2 \alpha_i$. Similarly for fixed i, at most one of $M_{i1}, \dots, M_{i(i-1)}, M_{i(i+1)}, \dots, M_{is}$ and M_i can have nonempty intersection with M. For $i < j$, let

$$N_{\{i,j\}} = \begin{cases} M_{ij} \cup M_{ji}, & \text{if } (M_{ij} \cup M_{ji}) \cap M = \varnothing \\ M_{ij} \cup M_j, & \text{if } M_{ji} \cap M \neq \varnothing \\ M_{ji} \cup M_i. & \text{if } M_{ij} \cap M \neq \varnothing \end{cases}$$

Then all of $N_{\{i,j\}}$'s are disjoint consisting of monomials absent in $\frac{G}{x_1^{(1)^{d-3}}}$, as $M_i \subset N_{\{i,j\}}$ and $M_i \subset N_{\{i,l\}}$ could not hold simultaneously if $j \neq l$.

Moreover, we have $|N_{\{i,j\}}| \geq$

$$\min \left\{ \frac{n_i n_j(n_i + n_j + 2)}{2}, \frac{n_j(3n_i(n_i+1) + (n_j+1)(n_j+2))}{6}, \frac{n_i(3n_j(n_j+1) + (n_i+1)(n_i+2))}{6} \right\},$$

which$\geq 2n_i n_j$ and the equality requires $(n_i, n_j) \in \{(1,1), (1,2), (2,1)\}$, and moreover $M_i \subset N_{\{i,j\}}$ if $n_i = 2$.

Therefore the number of monomials absent in $\frac{G}{x_1^{(1)^{d-3}}}$ is not smaller than $\sum_{i<j} |N_{\{i,j\}}| \geq 2\sum_{i<j} n_i n_j$. If the equality holds and there are some $n_i = 2$, then $s = 2$ and $n + 2 = 3$, which is impossible. So if the equality holds, we have $s = m$ and $n_1 = \cdots = n_s = 1$. In this case one of $x_1 x_2 x_3$ and $x_1 x_2 x_4$ must be absent. Therefore there are more than $2\sum_{i<j} n_i n_j$ terms absent from G.

ii) **The cases** A is unipotent. Then F is actually invariant[10] under A. Assume A is non-identity. Let J be the Jordan normal form of A, such that sizes of blocks are arranged in non-decreasing order. Then $J(x_i) = x_i + \varepsilon_i + x_{i+1}$ and $J(x_{n+2}) = J(x_{n+2})$, where $\varepsilon_i \in \{0, 1\}$ for $1 \leq i \leq n+1$. We say i is **regular** if $\varepsilon_s = 1$. Define a number $\alpha(J)$ by

$$\alpha(J) = \sum_{s \text{ regular}} \left(\binom{s+1}{2} + 1 \right).$$

Let us for a while assume the following two lemmas:

Lemma 3.2. $\alpha(J) > \operatorname{tr.deg}_k(A)$.

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[9] M_{ijl} is not used in this part.

[10] I think this is a little bit non-trivial? Anyway one can show all of eigenvalues of the induced action, by A on the space of homogeneous polynomials of degree d, equals 1.

Lemma 3.3. *Assume G is invariant under J, then coefficients of G satisfies at least $\alpha(J)$ linearly independent linear relations over k.*

Assuming there two lemmas, one can choose a matrix $P \in \mathrm{GL}_{n+2}\left(\overline{k(A)}\right)$ such that $PAP^{-1} = J$, and let $G(x) = F(P^{-1}(x))$. Then G is invariant under J.

Therefore

$$\mathrm{tr.deg}_k\{\text{coefficients of } F\} \leq \mathrm{tr.deg}_k\{\text{coefficients of } G\} + \mathrm{tr.deg}_k(A) < \binom{n+d+1}{d-1},$$

which contradicts to F is generic. □

Let us give the proofs of these two lemmas. To show Lemma 3.2 we need

Lemma 3.4. *Let $N \in M_{m\times m}(K)$ be a nilpotent matrix over K, $V_i = \mathrm{im}(N^i)$ and $\beta_i = \dim_K V_i$. Let $\beta(N) = 2\sum_{i=0}^{\infty}(\beta_i - \beta_{i+1})\beta_{i+1} = m^2 - \sum_{i=0}^{\infty}(\beta_i - \beta_{i+1})^2$. Then*

$$\mathrm{tr.deg}_k(N) \leq \beta(N).$$

Proof. Let Y be the subvariety of $\mathbb{A}_k^{m^2}$ consisting of nilpotent matrices N with $\dim \mathrm{im}(N^i) = \beta_i$. We only have to show $\dim_k(Y) = m^2 - \sum_{i=0}^{\infty}(\beta_i - \beta_{i+1})^2$.

As N is nilpotent, β_i's have determined the Jordan blocks of such N: there are exactly $\beta_{i-1} - \beta_i$ blocks of size not smaller than i. Let J be of such Jordan normal form. Then all such N lies in the orbit of J under the conjugating action of $\mathrm{GL}_m(k)$, thus

$$\dim Y = \dim \mathrm{GL}_m(k) - \dim C_{\mathrm{GL}_m(k)}(J) = m^2 - \sum_{i=0}^{\infty}(\beta_i - \beta_{i+1})^2.$$

□

Using this estimation we can easily prove Lemma 3.2.

Proof of Lemma 3.2. Assume $A = I_{n+2} + N'$, $J = I_{n+2} + N$. Using the above lemma, we have $\mathrm{tr.deg}_k(A) = \mathrm{tr.deg}_k(N') \leq \beta(N') = \beta(N)$. Then we only have to show $\alpha(J) > \beta(N)$. Let us divide into cases depending on regularities of $n-1, n$ and $n+1$. Note that since blocks are arranged in non-decreasing size and A is not identity, $n+1$ must be regular.

i) **The case** both $n-1$ and n are regular. Then

$$\alpha(J) \geq \binom{n+2}{2} + \binom{n+1}{2} + \binom{n}{2} + 3 > (n+2)^2 - (n+2) \geq \beta(N).$$

ii) **The case** $n-1$ is regular but n is not regular. Then all of sizes of blocks of N are not larger than 2, thus $N^2 = 0$, which implies $\beta(N) \leq \frac{(n+2)^2}{2}$. Thus

$$\alpha(J) \geq \binom{n+2}{2} + \binom{n}{2} + 2 > \frac{(n+2)^2}{2} \geq \beta(N).$$

iii) **The case** n is regular but $n-1$ is not regular. Then all of sizes of blocks of N are not larger than 3, thus $N^3 = 0$, which implies $\beta(N) \le \frac{2(n+2)^2}{3}$. Thus

$$\alpha(J) \ge \binom{n+2}{2} + \binom{n+1}{2} + 2 > \frac{2(n+2)^2}{3} \ge \beta(N).$$

iv) **The case** neither $n-1$ nor n is regular. Then $\beta_0 = n+2$, $\beta_1 = 1$ and $\beta_i = 0$ for $i \ge 2$. Thus $\beta(N) = 2n+2$. Therefore

$$\alpha(J) \ge \binom{n+2}{2} + 1 > 2n+2 = \beta(N).$$

□

To show Lemma 3.3 we have to choose special monomial to get coefficients that can be direct calculated under the transformation.

Proof of Lemma 3.3. Order the set of monomials of degree d lexicographically. If μ is the monomial $\prod_i x_i^{a_i}$, then

$$\mu(J(x)) = \mu(x) + \sum_{\nu<\mu} c_{\mu\nu}\nu$$

For some integers $c_{\mu\nu}$'s. Now we assume $G = \sum_\mu a_\mu \mu$, then

$$\sum_\mu a_\mu \mu = \sum_\mu a_\mu \mu + \sum_\nu \left(\sum_{\mu>\nu} c_{\mu\nu} a_\mu \right) \nu,$$

which implies $\sum_{\mu>\nu} c_{\mu\nu} a_\mu = 0$ for each ν.

So we just have to show $\mathrm{rk}((c_{\mu\nu})_{\mu,\nu}) \ge \alpha(J)$. As in under the lexicographic order this matrix is strictly lower-triangular, we only have to show there are at least $\alpha(J)$ monomials μ such that $c_{\mu\mu_-} \ne 0$, where μ_- is the predecessor monomial of μ under the lexicographic order.

We say a monomial is **regular** if it satisfies the above condition. To show there are at least $\alpha(J)$ regular monomials, let s be a regular index of J and choose a degree d monomial $\mu = x_{n+2}^{a_{n+2}} \prod_{i=1}^s x_i^{a_i}$ where $a_s > 0$. Then $\mu_- = \frac{x_{s+1}}{x_s}\mu$. One can directly calculate that $c_{\mu\mu_-} = a_s$.

If $\mathrm{char}(k)$ does not divide a_s then μ is regular. If we fix $a_s = 1$ and change other a_i's, there are $\binom{s+d-2}{d-1}$ different μ's. Moreover, we can also deduce that at least one of $x_s^2 x_{n+2}^{d-2}$ and $x_s^3 x_{n+2}^{d-3}$ is regular. Therefore there are at least $\binom{s+d-2}{d-1} + 1 \ge \binom{s+1}{2} + 1$ regular monomials of the above form. Summing up with respect to regular s's the lemma follows. □

4 Other Results

In [MM63], some corollaries was shown:

Corollary 4.1. *Let X be a degree-d smooth hypersurface in $\mathbb{P}^3$.*

a) If $d \ge 4$, X is a minimal model. Therefore $\mathrm{Bir}(X) = \mathrm{Aut}(X)$.

b) If $d = 4$ i.e. X is a K3 hypersurface, then $\mathrm{Aut}(X)$ *is discrete but can be infinite.*

Anyway, using today's theory of MMP it is no hard to show that

Proposition 4.2. *Let $n \geq 2$, $d \geq 3$ and X be a degree-d smooth hypersurface in $\mathbb{P}^{n+1}$, then X is a minimal model iff $d \geq n + 2$.*

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